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Measuring risk-aversion: The challenge



P.J. Thomas*

Safety Systems Research Centre, Queen's School of Engineering, University of Bristol, Queen's Building, University Walk, Clifton BS8 1TR, United Kingdom

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ABSTRACT

Risk-aversion is advanced as a measure of the feeling guiding the person who faces a decision with uncertain outcomes, whether about money or status or happiness or anything else of importance. The concepts of utility and, implicitly, risk-aversion were used first nearly 300 years ago, but risk-aversion was identified as a key dimensionless variable for explaining monetary decisions only in 1964. A single class of utility function with risk-aversion as sole parameter emerges when risk-aversion is regarded as a function of the present wealth, rather than subject to alteration through imagining possible future wealths. The adoption of a single class allows a more direct analysis of decisions, revealing shortcomings in the use of conventional, Taylor series expansions for inferring risk-aversion, over and above the obvious restrictions on perturbation size. Dimensional analysis shows that risk-aversion is a function of three dimensionless variables particular to the decision and a set of dimensionless character traits, identified later as the limiting reluctance to invest and the lower threshold on risk-aversion. The theoretical framework presented allows measurement of risk-aversion, paving the way for direct, evidence-based utility calculations.

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1. Introduction

Risk-aversion is a fundamental parameter determining how much satisfaction or utility we obtain from an experience, from a good or from money. It establishes the shape of the utility function that quantifies how much satisfaction or utility we derive. Before attempting to measure it, it is obviously necessary to know what risk-aversion is and have a proper model of how it acts through utility to affect people's decisions. The possibility then arises of using the decision as the measured parameter and inferring the risk-aversion that must have been in place for that decision to have been made.

Unfortunately there has been incomplete agreement in the past on the behaviour of risk-aversion, even on whether and when risk-aversion can be regarded as

constant. This lack of clarity has led to a number of different classes of functions being accorded the status of utility function, and this uncertainty has affected existing measurement methods, as will be explained. Thus the first challenge of this paper is to establish a realistic understanding of risk-aversion and its behaviour as a prelude to its accurate measurement.

The subject will be introduced by exploring the ways that the ideas of utility and risk-aversion have developed. The concepts are typically applied to the two canonical cases: the purchase of insurance and the purchase of a lottery ticket, which stand as proxies for additional types of decision under uncertainty. The conventional way of estimating the individual's risk-aversion in these cases uses a Taylor series expansion about the utility of the individual's starting wealth, as will be shown in Section 3. The required assumption of small deviations from that wealth brings with it significant limitations, but has the advantage that the form of the utility function does not need to be specified.

* Tel.: +44 117 331 5830; fax: +44 117 954 5208.

E-mail addresses: pjt3.michaelmas@gmail.com, philip.thomas@bristol.ac.uk.

A secondary advantage accrues in that a Taylor series expansion makes it unnecessary to consider whether or not the individual alters his risk-aversion when he is making a pairwise comparison between the utility of two possible outcomes. While there are good reasons for considering the risk-aversion of a wealthy person generally to be different from the risk-aversion of a poor person, it is argued in [1] that it is not feasible for a rich person considering the effect of a large insurance loss to experience the risk-aversion felt by a person subsisting at the post-loss level of wealth already. If it is accepted that risk-aversion is a feeling that develops from the experience of living in a given condition of life, lack of the required detailed knowledge and feel would prevent a rich person considering insurance thinking himself fully into the role of a poor person when evaluating his wealth after a substantial loss, even if he were minded to do so and however much he appealed to his imagination. Exactly the same argument applies to the poor man considering a lottery, and to people of all gradations of wealth in between. Nor is it sufficient to have lived in a different state of wealth in the past, since the feelings that the previous condition produced will be remembered so imperfectly that it would not be possible to develop the corresponding level of risk-aversion, even if the person wished so to do. A more realistic model is adopted in this paper. It is expected that the decision maker will vary his risk-aversion during the course of his pondering on his decision, but that risk-aversion will stay constant during each pairwise comparison of the outturn utilities resulting from the adoption or non-adoption of a particular course of action. This model is of strong economic importance, since it will be shown that the associated utility functions must then be of one class only, namely the Power utility, with risk-aversion as sole parameter.

Dimensional analysis will be used to clarify what can and should be measured in the insurance and lottery cases. Two additional, dimensionless parameters are recommended for measurement, which, like risk-aversion, are particular to the person. They are both limiting values: the first being the individual's limiting reluctance to invest (a scaled version of before and after utility differences) while the second is the individual's lower threshold on risk-aversion.

Worked examples will be given of possible measurement scenarios, both for lotteries and for insurance. Problems with the existing methods based on Taylor series expansions will be highlighted, including the rather striking fact that they fail to measure the right parameter in the case of insurance. Guidance will be given on how to obtain a good signal to noise ratio when measuring risk-aversion. Finally, the residual difficulties will be brought out of measuring a parameter that is personal to the individual, and that will vary according to the importance of the decision.

2. Development of the concepts of utility and risk-aversion

The study of utility as a way of explaining people's actions has a long and illustrious history, having gained

the attention of a series of distinguished scholars, from Daniel Bernoulli to John von Neumann. The derivatives of utility have a particular importance in economic theory. The first derivative will be discussed now and the second derivative later in this Section. The first derivative, known to economists as 'marginal utility', is defined by the Encyclopaedia Britannica [2] as "the additional satisfaction or benefit (utility) that a consumer derives from buying an additional unit of a commodity or service". The utility to a consumer of an additional unit of a product is normally taken to be inversely related to the number of units of that product he already owns.

The concept of marginal utility was key to Jevons's solution [3] of the 'paradox of value', which had perplexed economists until the late 19th century and is illustrated in the much higher monetary value attached to diamonds as compared with the same mass of bread, even though the latter is an important dietary component and the former merely an adornment. How can this be? Marginal utility allowed the following explanation. People are attracted to diamonds but the fact that they are scarce means that only a small number of people can have many of them. Under these conditions the marginal utility of diamonds to those people with few of them will be high, which explains why they command a high price. On the other hand, bread is in plentiful supply, so that customers for bread can soon possess enough to satisfy their most pressing need. As a person's appetite for bread becomes satisfied, so the additional utility of a further slice, the marginal utility, will go down, with the result the price he will be prepared to pay will fall. A glut of bread could drive its price down to practically zero, since all or almost all potential customers would have enough bread already [2,4].

Despite the success of marginal utility in providing a conceptual framework for understanding the paradox of value, there was clear difficulty in measuring quantitatively the utility that a person received from consuming a product. So shortly after Jevons's work, Edgeworth [5] began the development of indifference theory. Whereas utility theory assumes, at least in principle, the numerical measurability of the difference in the utility conferred by two options on a person or an organisation, indifference theory rests on the weaker assumption that the person can specify only which option yields him the higher utility. Further important work, carried out by Pareto in the 1900s [6] and, thirty years later, by Hicks [7] led to the wide acceptance of indifference theory.

But shortly after Hicks' *magnum opus* was published, the possibility of numerical measurement of utility was revived by von Neumann and Morgenstern in their highly influential book, *Theory of Games and Economic Behaviour* [8], published in 1944. In a direct challenge to Hicks's indifference curve methods, they claimed that:

"the treatment by indifference curves implies either too much or too little: if the preferences of the individual are not all comparable, then the indifference curves do not exist. If the individual's preferences are all comparable, then we can even obtain a (uniquely defined) numerical utility, which renders the indifference curves superfluous."

Von Neumann was astonished that the approach had not been pursued before: “Ja hat denn das niemand gesehen?” (“But has nobody seen this before?”) was said to have been his exclamation after writing down a set of axioms to underpin expected utility theory [9]. Von Neumann and Morgenstern pointed out that, for the case where options, A, B, and C were put in that order of preference by an individual, then a numerical measure of utility could be obtained by eliciting from him a further piece of information, namely the probability, α , at which he would be prepared to accept a probabilistic combination of A and C as equivalent to option B. The individual's indifference between options B and the probabilistic combination of options A and C produces an equality in utility:

$$u(B) = \alpha u(A) + (1 - \alpha)u(C) \quad (1)$$

where $u(\cdot)$ is the utility function used to calculate the person's utility from the option. Eq. (1) provides a schema for allocating a utility value to option B, given the utility of options A and C, and has come to be known as the ‘standard gamble’.

The book by von Neumann and Morgenstern introduced Game Theory as a new field of study. It also put the definition of utility theory on an axiomatic basis, proving that if a numerical measurement or estimate of a person's utility, u , could be made, then that utility would be correct to a positive linear transformation, so that v would be an equally valid measure of utility, where $v = au + b$; $a > 0$.

But utility functions had been used well before this time, with the Swiss mathematician, Gabriel Cramer incorporating a utility function in his proposed solution to the ‘St. Petersburg Paradox’. This was a gambling problem where the mathematical expectation of money won gave poor guidance to the decision maker [10], and a brief discussion is offered because of its importance to the development of expected utility theory, a concept of relevance to this paper. The game consists of counting the tosses of a fair coin before a head appears and paying out the sum of $\text{£}2^{n-1}$ when a head occurs for the first time on the n th toss. The question arises as to what is a fair price to pay for the opportunity to play this game. The head could occur on the first throw or on the second or, with decreasing probability, on the third or fourth and so on. The chance of a sequence of $n - 1$ tails followed by a head is $(1/2)^{n-1} \times 1/2 = 1/2^n$. A return must come at some point, with a low pay out if the first head turns up early and a high return if it occurs late. Allowing for all possible outcomes, the expected value of that random monetary gain, G , is $E(G) = \sum_{n=1}^{\infty} (1/2^n)2^{n-1} = 1/2 \sum_{n=1}^{\infty} (1/2^n)2^n = 1/2(1 + 1 + 1 + \dots) = \infty$. Using this result for guidance, a person should be prepared to pay an enormous sum for the chance to play, but the paradox lies in the fact that no-one is prepared to pay very much at all. Cramer suggested in a 1728 letter to Nicolas Bernoulli (included at the end of [11]) that “in their theory, mathematicians evaluate money in proportion to its quantity while, in practice, people with common sense evaluate money in proportion to the utility they can obtain from it.” To model this ‘common sense’ evaluation, he used a square root function to generate the utility of the gains in his attempt to explain how normal people would react to the

problem. In finding the utility of the gain rather than of the individual's total wealth he was anticipating Kahneman and Tversky's Nobel prize winning work of 250 years later [12]. The same concept has been used recently to explain multibuy retail promotions such as buy-one-get-one-free [13–15].

Rather than the expected value, $E[G]$ of the random gain, G , Cramer now found the expected value of utility, $E[U]$, in which $U = U(G)$:

$$\begin{aligned} E[U] &= \sum_{n=1}^{\infty} \frac{1}{2^n} \left(2^{n-1}\right)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{2^n} 2^{\frac{n-1}{2}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^n \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^{n-1} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^k \\ &= \frac{1}{2} \left(1 + \left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^3 + \dots\right) = \frac{1}{2 - \sqrt{2}} \end{aligned} \quad (2)$$

where the fact that $\sqrt{2}/2$ is less than 1.0 causes the series to converge and so produce the value shown at the end. The monetary sum corresponding to this level of utility of gain is now found by reversing the square root transformation, viz. taking the square: $(E(U))^2 = \text{£}2.91$, to give the certainty equivalent, which Cramer suggested would be reasonable for a person to pay to play the game. The smallness of this sum suggests that a ‘man with common sense’ would be much too risk averse to be guided by the expected value of gain alone.

A similar answer was proposed independently by Daniel Bernoulli in 1738 [11], who made two changes, however. The first was to use a logarithmic utility function, and the second was to argue for the inclusion of the wealth of the player as an important influence. Having paid a sum, t , for a ticket to play the game, the player's expected change in utility from that, $u(w)$, of his initial wealth, w , would be

$$\begin{aligned} E[U] - u(w) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \ln(w - t + 2^{n-1}) - \ln w \\ &= \frac{1}{2} \ln(w - t + 1) + \frac{1}{4} \ln(w - t + 2) \\ &\quad + \dots + \frac{1}{2^k} \ln(w - t + 2^{k-1}) + \dots - \ln w \\ &= \ln \left((w - t + 1)^{\frac{1}{2}} (w - t + 2)^{\frac{1}{4}} \dots (w - t + 2^{k-1})^{\frac{1}{2^k}} \dots \right) - \ln w \end{aligned} \quad (3)$$

The player will be indifferent between playing or not playing the game if the expected utility from playing is the same as the starting utility, implying that

$$(w - t + 1)^{\frac{1}{2}} (w - t + 2)^{\frac{1}{4}} \dots (w - t + 2^{k-1})^{\frac{1}{2^k}} \dots - w = 0 \quad (4)$$

Now $(a + b)^{1/(2b)} = (1 + a/b)^{1/(2b)} b^{1/(2b)} = e^{(\ln(1+a/b))/(2b)} e^{(\ln b)/(2b)}$, so that for constant a , $\lim_{b \rightarrow \infty} (a + b)^{1/(2b)} = 1$.

Putting $a = w - t$ and $b = 2^{k-1}$, it can be seen that each of the product terms in Eq. (4), including the final, k^{th} term, is bounded. In practice, the terms in the product on the left-hand side of Eq. (4) are very close to unity from about

$k = 25$ onwards, so that a highly accurate, implicit solution may be found by using 50 terms. Thus a person with a total starting wealth of £3 should be prepared to pay up to £2.54, one with a £1000 should pay up to £5.97, while a millionaire should be prepared to pay only a little more at £10.94. (In fact, as pointed out by Bernoulli, for large wealths, the approximation $t \approx 0$ can be used in Eq. (4). More precisely, $w - t$ is being replaced by w , indicating that the ticket-price has been deducted already from the starting wealth, so that prior ownership of the ticket is assumed. Hence what is then being found from Eq. (4) is the ‘certainty equivalent’ gain in wealth from holding a ticket to play.) Thus Bernoulli provided a second mathematical justification for the common sense view that only small sums should be risked on this game, a solution he acknowledged freely was similar to Cramer’s. His paper was further noteworthy in defining the expected utility (“*emolumentum medium*”) when many options are possible, and the use of expected utility in insurance, where he explained quantitatively the different viewpoints of the insurer and the insured because of their difference in wealth.

Daniel Bernoulli’s 1738 paper defined marginal utility, $m(w)$, as a function of wealth, and in differential terms: “the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed”, a statement that is almost identical to the last line of the first paragraph of this Section. The only change of significance is the replacement of Bernoulli’s “proportionate” by the more general “related”.

The derivative of marginal utility has particular importance for risk studies. Pratt [16] made a link between risk and the negative of the normalised derivative of marginal utility, calling it the “local proportional risk aversion”. ‘Elasticity’ is the word used by economists to denote a normalised derivative, so that the same quantity can also be described as the ‘negative of the elasticity of marginal utility of wealth’. The ‘coefficient of relative risk aversion’ is another name. But the desirability of a shorter title means has led the author to use the hyphenated term, ‘risk-aversion’, ε , here and elsewhere:

$$\varepsilon = -\frac{w}{m} \frac{dm}{dw} = -w \frac{u''}{u'} \quad (5)$$

Risk-aversion, ε , as defined in Eq. (5), is a measure of a person’s actual aversion to risk (a demonstration of the validity of this notion is given at the end of Section 3.1). Regarding aversion to risk as a feeling or an attitude (which we will hope to measure under certain circumstances), ε is helped enormously as a mathematical measure of aversion to risk by its non-dimensional status, which allows it to be applied universally. This is necessary because the feeling of unease in the presence of uncertainty that constitutes aversion to risk may occur in a multiplicity of contexts: when considering dangers to health, to life style, to status, and so on, as well as to wealth. For an example quite unrelated to money see [17], where the apparently anomalous behaviour of 5-year old children at play may be explained quantitatively using risk-aversion.

It may be commented that lack of universality is a severe handicap to an alternative sometimes advanced as a

measure of aversion to risk, namely $-u''/u'$. While this expression might seem to possess a superficial generality as the ratio of two derivatives, and is dignified by the name ‘coefficient of absolute risk aversion’, which seems to have a fundamental ring, we may see from Eq. (5) that this parameter is in fact simply ε/w . Hence, if we take w to be wealth, for example, the parameter will have units £^{-1} or ¥^{-1} or $\text{\$}^{-1}$ or whatever. Thus even in the straightforward case of decisions regarding wealth, a single currency will need to be agreed if the same value is to be returned in fully analogous circumstances (see also [18]). But now let us take a diverse case, where, for example, w denotes popularity that may be at risk from a person’s decision. It is clear that there will be enormous if not insuperable difficulties in comparing the coefficient of absolute risk aversion applicable in this situation with the corresponding value in the case where wealth is at stake. In particular, this will be the case even when the person’s levels of stress hormones are very similar when he is taking the two decisions, which would indicate that he was feeling a similar aversion to risk.

The interpretation of risk-aversion in terms of what a person feels when taking a decision, irrespective of the precise subject of that decision, gives guidance on how that parameter may vary during the decision making process. For when he is pondering on his decision between two options where the balance of advantage is not clear cut, the person explores in his mind first how he would need to feel to choose option A and all that it entails, and then how he would need to feel to choose option B and its likely consequences. This process may be seen as the decision maker trying out various levels of risk-aversion to find the one with which he feels most content.

But, as explained in the Introduction, he will maintain his risk-aversion constant during each pairwise comparison of outturn utilities.

This conclusion has significant implications for the permissible form for a utility function, which Section 5 shows must belong to the Power family of utility functions, expressed most generally as:

$$u(w) = \begin{cases} \frac{w^{1-\varepsilon} - 1}{1 - \varepsilon} & \text{for } \varepsilon \neq 1 \\ \ln w & \text{for } \varepsilon = 1 \end{cases} \quad (6)$$

Interestingly, it may be seen from Eq. (6) that the logarithmic utility function advocated by Bernoulli corresponds to a utility function with a risk-aversion of 1.0. This is, in fact, the value of risk-aversion recommended by the UK’s Treasury [19]. Meanwhile, Cramer’s square-root utility function can be seen to imply a risk-aversion of 0.5. Thus both Bernoulli and Cramer chose to use the Power utility function, and their results bracket a recent estimate of the average risk-aversion of UK citizens, $\varepsilon = 0.85$ [20,21]. It should be noted, however, that the range of risk-aversions considered during the period of mulling over the decision may be much wider, and the value of risk-aversion finally selected might well be different from either Bernoulli’s or Cramer’s suggestion, depending on the importance of the decision to the person concerned, a question tied up intimately with his level of wealth.

3. Measurement of risk-aversion based on Taylor series expansions

By the argument made in Section 1, only the Power family of utility functions, with risk-aversion as sole parameter, can offer a truly realistic model of human decision making. In fact, economists and actuaries often adopt the Power utility function [22]. Nevertheless others have chosen different functions of roughly the same shape to represent utility, and this has provided an incentive to devise methods of measuring risk-aversion independent of the precise form of the utility function used. Such methods have typically applied a Taylor series expansion to the two canonical cases of a decision under risk, namely an individual taking part in a lottery or an individual seeking insurance. The small perturbations permitted by these methods restrict their range of validity, of course, but additional shortcomings have been found to exist.

3.1. Where the individual takes a voluntary risk: a lottery

Let there be a lottery offering a prize, z , with probability of winning, p . In such a case, the expected value of the random payout, Z , to the individual (either z or 0) will be

$$E[Z] = pz + (1 - p) \cdot 0 = pz \quad (7)$$

If the individual's maximum acceptable price (MAP) for a ticket is $t_0 = pz$, it can be seen that he is valuing the ticket on its expected money value only: $t_0 = E[Z]$. This is the risk-neutral position, and corresponds to a risk-aversion of zero, a statement justified at the end of this subsection. But the individual may be prepared to pay a different amount, indicating a different level of risk-aversion. It is reasonable to assume that the amount, t , that an individual of starting wealth, w , pays for a ticket gives an indication of his dimensionless risk-aversion, ε .

Buying a ticket for t will mean that his wealth decreases initially from w to $w - t$, which will mean that the utility of his wealth, its worth to him, will also decrease. That decrease will be permanent if he fails to win, when his utility will be $u(w - t)$. This situation has a probability of occurrence of $(1 - p)$. On the other hand, if he wins then his wealth will rise to $w - t + z$, bringing with it an increase in utility to $u(w + z - t)$, which has a probability, p , of being valid.

The person's expected utility after buying a ticket is then:

$$E[U] = pu(w + z - t) + (1 - p)u(w - t) \quad (8)$$

Assuming that both $z - t \ll w$ and $t \ll w$, each of the two utility terms on the right-hand side of Eq. (1) may be expanded using the first three terms of a Taylor series:

$$u(w + z - t) \approx u(w) + (z - t)u'(w) + \frac{1}{2}(z - t)^2u''(w) \quad (9)$$

$$u(w - t) \approx u(w) - tu'(w) + \frac{1}{2}t^2u''(w) \quad (10)$$

The warning needs to be given at this point that requirement that $z - t \ll w$ means that lotteries offering a very large prize, such as a national lottery, are not covered by this analysis.

Substituting into Eq. (8) gives:

$$\begin{aligned} E[U] &= pu(w) + pzu'(w) - ptu'(w) + \frac{1}{2}pz^2u''(w) - pztu''(w) \\ &\quad + \frac{1}{2}pt^2u''(w) + u(w) - tu'(w) + \frac{1}{2}t^2u''(w) - pu(w) \\ &\quad + ptu'(w) - \frac{1}{2}pt^2u''(w) = pzu'(w) + \frac{1}{2}pz^2u''(w) \\ &\quad - pztu''(w) + u(w) - tu'(w) + \frac{1}{2}t^2u''(w) = u(w) \\ &\quad + (pz - t)u'(w) + \frac{1}{2}(pz^2 - 2pzt + t^2)u''(w) \end{aligned} \quad (11)$$

The individual will expect to see an advantage or at least no disadvantage from buying a lottery ticket. Therefore a price acceptable to an individual is one that will cause him to expect the utility after buying the ticket to be at least as great as his utility without buying it:

$$E[U] \geq u(w) \quad (12)$$

Combining inequality (12) with Eq. (11) implies that:

$$(pz - t)u'(w) + \frac{1}{2}(pz^2 - 2pzt + t^2)u''(w) \geq 0 \quad (13)$$

which may be rearranged into the form:

$$-\frac{u''(w)}{u'(w)} \leq \frac{2(pz - t)}{pz^2 - 2pzt + t^2} \quad (14)$$

subject to the important provisos that

(i) $u'(w) > 0$, and

$$(ii) \quad pz^2 - 2pzt + t^2 = z^2 \left(p \left(1 - 2\frac{t}{z} \right) + \left(\frac{t}{z} \right)^2 \right) > 0 \quad (15)$$

Condition (i) can be assumed to hold since utility can be expected to increase with wealth, while condition (ii) will be valid provided $t \leq z/2$, which can be expected to be the case for most lotteries of interest.

Using the definition of risk-aversion, ε , given by Eq. (5), the value of risk-aversion, ε , corresponding to a fixed ticket price, t , will obey the inequality:

$$\varepsilon \leq \frac{2w(pz - t)}{pz^2 - 2pzt + t^2} \quad (16)$$

If tickets are on sale at a fixed price and bought by a number of people, the highest value of risk-aversion that an individual with wealth, w_i , may experience while still being prepared to buy the ticket for t is given by the equality condition of inequality (16):

$$\varepsilon(\max|t, w_i) = \frac{2w_i(pz - t)}{pz^2 - 2pzt + t^2} \quad (17)$$

An individual with wealth, w_i , might have the potential to be more risk confident in this situation, so that his risk-aversion is less than $\varepsilon(\max|t, w_i)$ and be prepared to offer

more than t for the ticket, but is not being pushed to this level. If so he is benefiting from what is known by economists as the 'consumer surplus' that may be obtained in free and open markets where one price is offered to all.

But it is assumed at this point that the person's maximum acceptable price, $t_{\max i}$, for a lottery ticket may be elicited, perhaps by some sort of auction. The process of pushing the individual to his maximum price can be visualised as one where the general price of the ticket is raised, which will affect $\varepsilon(\max|t, w_i)$, and recording how long the individual stays in the auction. Now $\varepsilon(\max|t, w_i)$ is a decreasing function of t as long as t lies in the range $0 < t < t_0(1 + \sqrt{o_A})$, where $o_A = \sqrt{(1-p)/p}$ is the odds against winning (see Appendix A). Since p is small in a typical lottery, the upper bound of the range is likely to be large. Thus raising the price has the effect of 'lowering the bar' on maximum allowable risk-aversion, $\varepsilon(\max|t, w_i)$. To stay in, the person must be able to tolerate a lower risk-aversion (be bolder) than the current level of $\varepsilon(\max|t, w_i)$. The further important assumption is now made that when person i has been pushed to offer his MAP, $t_{\max i}$, his expected utility after buying the ticket will just match his utility without buying it – the 'utility break-even' condition. This assumption is often made without comment as if obviously true, but in fact its justification requires the operation of a further mechanism, as explained in Section 8.

Assuming the operation of the mechanism described in Section 8, the individual's risk-aversion, ε_{BEi} , at this break-even condition now follows from the development of the equality condition of inequality (12): $E[U] = u(w)$, leading to:

$$\varepsilon_{BEi} = \frac{2w_i(pz - t_{\max i})}{pz^2 - 2pz t_{\max i} + t_{\max i}^2} = \frac{2w_i/t_{\max i}(pz/t_{\max i} - 1)}{p(z/t_{\max i})^2 - 2pz/t_{\max i} + 1} \quad (18)$$

Eq. (18) has an auxiliary purpose in that it may be used to justify the use of ε as a measure of aversion to risk. For if the individual's maximum acceptable price for a ticket happens to be $t_{\max i} = pz$, then Eq. (18) gives $\varepsilon_{BEi} = 0$. This position, where the individual's maximum acceptable price is set by purely monetary considerations (substitute $\varepsilon = 0$ into Eq. (6)) may be regarded as the risk neutral position. On the other hand, should person i set $t_{\max i} < pz$ then $\varepsilon_{BEi} > 0$. Such a person can be seen to be exhibiting greater caution with respect to the lottery than the person who sets $t_{\max i} = pz$, and therefore might reasonably be called risk averse. But a person setting $t_{\max i} > pz$ will be keener to play the lottery than the person who sets $t_{\max i} = pz$ and may thus be described as risk confident. Now $\varepsilon_{BEi} > 0$. Hence ε is well correlated with what we might in normal speech call aversion to risk, which provides a justification for calling it 'risk-aversion'.

3.2. When the individual seeks to avoid an already-imposed risk: insurance

Now let us consider an individual of starting wealth, w , facing a loss z with probability, p , in a given period and use

his insurance premium, t , for avoiding any possibility of this loss to estimate his risk-aversion, ε .

If he does not buy insurance, the expected value, $E[U_{\text{end}}]$, of his utility at the end of the period, U_{end} , will be:

$$E[U_{\text{end}}] = pu(w - z) + (1 - p)u(w) \quad (19)$$

whereas if he buys the insurance at premium, t , his utility at the end of the period will be $u(w - t)$, with certainty. The individual will be content to pay a premium, t , satisfying:

$$u(w - t) \geq pu(w - z) + (1 - p)u(w) \quad (20)$$

An expansion of the term, $u(w - t)$, using the first two terms of a Taylor series, has been provided above as Eq. (10), under the assumption that $t \ll w$. The term, $u(w - z)$ may be expanded in a similar fashion, provided it is possible to assume that $z \ll w$:

$$u(w - z) \approx u(w) - zu'(w) + \frac{1}{2}z^2u''(w) \quad (21)$$

The necessity of assuming that $z \ll w$ means that this analysis can apply to the insurance of, for example a washing machine costing £250, with an annual insurance premium of tens of pounds, but not directly to house insurance for the average person. This point will be returned to later.

Substituting from Eqs. (10) and (21) into inequality (20) gives:

$$u(w) - tu'(w) + \frac{1}{2}t^2u''(w) \geq pu(w) - pzu'(w) + \frac{1}{2}pz^2u''(w) + u(w) - pu(w) \quad (22)$$

or

$$\frac{1}{2}u''(w)(t^2 - pz^2) \geq (t - pz)u'(w) \quad (23)$$

Hence

$$u''(w) \left(\left(\frac{t}{z} \right)^2 - p \right) \geq \frac{2}{z} \left(\frac{t}{z} - p \right) u'(w) \quad (24)$$

It is safe to assume that both the probability of loss, p , and the ratio of the maximum acceptable premium to the loss, t/z , are both strictly fractional. Fractionality implies the following two inequalities:

$$\left(\frac{t}{z} \right)^2 < \frac{t}{z} \quad (25)$$

and

$$p < p^{\frac{1}{2}} \quad (26)$$

Now the expected loss is given by

$$E[Z] = pz + (1 - p) \cdot 0 = pz \quad (27)$$

(c.f. Eq. (7)). In the (unlikely) case where the premium is set less than or equal to the expected loss, then $t \leq pz$ or $t/z \leq p$ so that, using inequality (25), $(t/z)^2 < p$ or $(t/z)^2 - p < 0$. Moreover, when the premium is set above the expected loss but in the range: $p < t/z < p^{\frac{1}{2}}$ (a range of more than an order of magnitude when $p < 0.01$), the same condition still holds: $(t/z)^2 - p < 0$. The reasonable

assumption is made that $u'(w) > 0$, corresponding to utility increasing with wealth. Moreover, the term, $\left(\left(\frac{t}{z}\right)^2 - p\right)$, will be negative whenever $t/z < \sqrt{p}$. Hence, dividing both sides of the inequality (24) by these terms will give:

$$\frac{u''(w)}{u'(w)} \leq \frac{2}{z} \frac{\frac{t}{z} - p}{\left(\frac{t}{z}\right)^2 - p} \quad \text{for } \frac{t}{z} < \sqrt{p} \quad (28)$$

or

$$\frac{u''(w)}{u'(w)} \leq \frac{2}{z} \frac{p - \frac{t}{z}}{p - \left(\frac{t}{z}\right)^2} \quad \text{for } \frac{t}{z} < \sqrt{p} \quad (29)$$

Hence

$$-\frac{u''(w)}{u'(w)} \geq \frac{2}{z} \frac{p - \frac{t}{z}}{\left(\frac{t}{z}\right)^2 - p} \quad \text{for } \frac{t}{z} < \sqrt{p} \quad (30)$$

Hence, using the definition of risk-aversion from Eq. (5) and rearranging gives the boundary on risk-aversion set by a fixed premium, t :

$$\varepsilon \geq \frac{2w(pz - t)}{(t^2 - pz^2)} \quad \text{for } \frac{t}{z} < \sqrt{p} \quad (31)$$

If a common premium, t , is set and taken up by a number of people, each with wealth, w_i , then the lowest value of risk-aversion that an individual with wealth, w_i , may experience while still being prepared to pay an insurance premium of t is given by the equality condition of inequality (31):

$$\varepsilon(\min|w_i, t) = \frac{2w_i(pz - t)}{t^2 - pz^2} \quad (32)$$

Any individual, i , may have the potential to be more risk-averse in this situation than is suggested by $\varepsilon(\min|w_i, t)$ because he is temperamentally more timid, but the premium level is not pushing him to his maximum caution or highest level of risk-aversion. So in the case of insurance, too, it is possible for an individual to benefit from the 'consumer surplus' noted with a lottery.

In a similar way to the lottery case, the method now involves the assumption that the person's maximum acceptable price, $t_{\max i}$, for the insurance premium may be elicited.

The process of pushing the individual to the maximum premium he will accept can be visualised as one where the general insurance premium is raised, which will affect $\varepsilon(\min|t, w_i)$, and recording how long the individual stays in the exercise. Now $\varepsilon(\min|t, w_i)$ is an increasing function of t as long as t lies in the range $0 < t < z\sqrt{p}$. Thus raising the premium has the effect of 'raising the bar' of minimum allowable risk-aversion, $\varepsilon(\min|t, w_i)$. To stay in, the person must be able to tolerate a higher risk-aversion (be more cautious) than the current level of $\varepsilon(\min|t, w_i)$.

The further important assumption is now made that when person i has been pushed to offer his MAP, $t_{\max i}$, his expected utility after buying the insurance will just match his utility without buying it – the 'utility breakeven' condition. Thus his risk-aversion, ε_{BEi} , at this breakeven condition now follows from the development of the equality condition of inequality (31), leading to:

$$\varepsilon_{BEi} = \frac{2w_i/t_{\max i}(pz/t_{\max i} - 1)}{\left(1 - p(z/t_{\max i})^2\right)} \quad \text{for } \frac{t_{\max i}}{z} < \sqrt{p} \quad (33)$$

The assumption that the person decides on his maximum acceptable price for insurance, $t_{\max i}$, at the utility break-even condition is often taken for granted, but this will be considered further in Section 6, which shows that this assumption can be improved upon.

3.3. The nature of the minimum and maximum risk-aversions for person i

It may be observed first that a person's limiting value of risk aversion depends on his starting wealth, w_i , as well as on the particular circumstances of the decision: the amount to be gained or lost and the probability of either eventuality. Part of the dependence on wealth is explicit in Eqs. (18) and (33), but a further important dependency comes in via the individual's choice of maximum ticket price or maximum insurance premium, $t_{\max i}$. These will be affected by how much money the person has: $t_{\max i} = t_{\max i}(w_i)$, as asserted by Bernoulli [11]. So while a casual reading of Eqs. (18) and (33) might suggest that risk-aversion would rise with wealth, in fact the reverse is found to be true normally, a fact that is explicable in the two cases considered above in terms of the variability of $t_{\max i}$ with starting wealth. For example, a poor person might decide to insure a washing machine for more than the expected loss, indicating a positive risk-aversion, while a rich person might decide that he can stand the loss and pay nothing for insurance. Putting $t_{\max i} = 0$ into Eq. (33) for the case of the rich person shows that his minimum risk-aversion is negative in this case, and less than the poor person whose choice is to take out insurance because he cannot stand the loss.

In considering the most he will pay to take part in a lottery or to be insured, the person may consider several maximum ticket prices or maximum insurance premia, $t_{\max i}^{(test)}$, as candidates for his $t_{\max i}$ and carry out pairwise utility comparisons, for example between the outturns with and without insurance at that premium. The possible maximum price or premium, $t_{\max i}^{(test)}$, will change before each successive comparison of outturns – this is part of the process of pondering on a decision. But risk-aversion will stay constant during the comparison process itself, a procedure endorsed implicitly by Bernoulli, who, by choosing a logarithmic utility function, kept the risk-aversion constant at $\varepsilon = 1.0$. See Eq. (3), where the same utility function, $u(\cdot) = \ln(\cdot)$, is applied whether the wealth is low: $w - t + 1$ or extraordinarily high: $w - t + 2^{k-1}$, $k \rightarrow \infty$.

4. Using dimensional analysis to shed light on risk-aversion in both insurance and lottery cases

4.1. Insurance premium

As in Section 3.2, let it be assumed that the person's maximum acceptable price for the insurance premium can be elicited in some reliable way. It is then reasonable to expect the risk-aversion for person i , $\varepsilon_{t_{\max i}}$, buying an

insurance premium at his maximum acceptable price to depend on the maximum the person is prepared to pay for his insurance premium (£), $t_{\max i}$, and on two further sets of variables. The first set will be those variables particular to the decision currently being made, namely:

- z , the value of the possible loss (£)
- w_i , the person's initial wealth (£)
- p , the probability of incurring the loss (dimensionless).

while the second set will be the minimum number, m , of underlying character traits, $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, relevant to taking decisions under risk and particular to the individual. The influence of these traits on the decision will be expressed via risk-aversion, ε_i , the value of which they may influence by one of the following operations:

- increase, by addition of a positive number or by multiplication by a number greater than unity,
- decrease, by subtraction of a positive number or by division by a number greater than unity, or
- limit, through setting a limiting value.

If the nondimensionality of risk-aversion is to be preserved, the parameters, τ_{ki} , involved in such operations must be dimensionless. Moreover, if one of the parameters, τ_{ki} , should depend on other parameters, then the requirement for the nondimensionality of risk-aversion requires that relationship to be expressible as a function of nondimensional variables. Moreover, the parameters, τ_{ki} , are, by definition, independent of the problem being considered, and so will be constant for any given decision. A set of suitable parameters, τ_{ki} , will be derived in Section 6, but it is not necessary at this stage either to specify their number, m , or what they are.

Observe that the notation for risk-aversion has been changed from ε_{BEi} to $\varepsilon_{t\max i}$, which denotes the risk-aversion associated with individual i 's maximum acceptable premium, not tying it to the condition of utility break-even. This form removes the assumption implicit in the Taylor series approach that the maximum acceptable price for the insurance premium occurs at the condition of utility breakeven.

The $m + 4$ variables just discussed are advanced as complete in the sense that no other influence is significant, so that we may write:

$$\varepsilon_{t\max i} = f(t_{\max i}, z, w_i, p, \tau_{1i}, \tau_{2i}, \dots, \tau_{mi}) \quad (34)$$

Using square brackets, $[\cdot]$, to denote the operation of finding the dimension of a parameter, we may write

$$\begin{aligned} [\varepsilon_{t\max i}] &= 1 \\ [t_{\max i}] &= C \\ [z] &= C \\ [w_i] &= C \\ [p] &= 1 \\ [\tau_{ki}] &= 1, \quad k = 1, 2, \dots, m \end{aligned} \quad (35)$$

where C represents currency, while 1 denotes non-dimensionality. Clearly only one dimension is displayed

in the $m + 4$ variable set listed on the right-hand side of Eq. (34), and this is currency. Applying the principles of dimensional analysis [23–25], we may non-dimensionalise by choosing the single parameter, $t_{\max i}$, as the sole member of a dimensionally independent subset. (Since there is only one member, it satisfies trivially the condition that its dimension cannot be composed of the dimensions of different members.) This subset is dimensionally complete in the sense that the dimensions of remaining the variables, namely $\varepsilon_{t\max i}, z, w_i, p$, and τ_{ki} , $k = 1, 2, \dots, m$ may be expressed as powers (including a power 0 as necessary) of the dimensions of that subset, in this case the single dimension of currency, C .

By the Buckingham Pi Theorem [23–25], the number of independent quantities governing the individual's minimum risk-aversion is then reduced from $m + 4$ to $m + 3$, and the dimensionless parameter, $\varepsilon_{t\max i}$, may specified in terms of dimensionless quantities:

$$\varepsilon_{t\max i} = g\left(\frac{z}{t_{\max i}}, \frac{w_i}{t_{\max i}}, p, \tau_{1i}, \tau_{2i}, \dots, \tau_{mi}\right) \quad (36)$$

Thus the variables particular to the problem may be reduced to: (i) the size of the possible loss, denominated in units of the individual's maximum acceptable insurance premium, $z/t_{\max i}$, (ii) the size of the individual's wealth, denominated in the same units, $w_i/t_{\max i}$ and (iii) the probability of the loss, p . The probability of loss, p , its magnitude, z , and the individual's starting wealth, w_i , will all be constant for a particular decision, as will the personal traits, $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, by definition constant over long periods. Hence it is clear from Eq. (36) that $\varepsilon_{t\max i}$ can vary only as a result of the variation in $t_{\max i}$ as different values are selected for consideration.

[The formulation of Eq. (36) is not unique, since it would be possible to choose either z or w_i as the sole member of a dimensionally independent subset. Choosing the latter as an example would result in the first two variables in Eq. (36) being changed to z/w_i and $t_{\max i}/w_i$, for some new function, g . Of course, the fact that we may write $z/w_i = (z/t_{\max i})(w_i/t_{\max i})^{-1}$ and $t_{\max i}/w_i = (w_i/t_{\max i})^{-1}$ shows the equivalence of the two approaches.]

By the model of [1], the individual will postulate to himself several test maximum insurance premia, $t_{\max i}^{(test)}$, during the process of pondering his decision, each of which will have associated with it a corresponding risk-aversion, $\varepsilon_{t\max i}^{(test)}$, given by Eq. (36). A pairwise comparison will be carried out at each test value, $t_{\max i}^{(test)}$, between the utility following the purchase of insurance at that premium, $t_{\max i}^{(test)}$, and the expected utility without, using the same test risk-aversion, $\varepsilon_{t\max i}^{(test)}$.

The model of [1] implies that associated with each test premium, $t_{\max i}^{(test)}$, there will be a unique value of risk-aversion, $\varepsilon_{t\max i}^{(test)}$, which will be used in the calculation of utility at the three possible wealth states associated with the decision: w , $w - z$ and $w - t_{\max i}^{(test)}$. Although it shows that risk-aversion must be a constant at each value of the test premium, Eq. (36) does not prove this premise, since there could conceivably be three different functions, g_1 , g_2 and g_3 , satisfying Eq. (36), producing a different

risk-aversion at each wealth level. Neither does Eq. (36) provide a confirmation of Taylor-series Eq. (33), nor vice versa. However the two equations do identify the same three independent variables particular to the current decision. The neglect in Eq. (33) of the individual's personal traits, $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, implies the assumption in the Taylor series model that the individual's risk tendencies, either aversion to risk or confidence in the face of risk, are captured fully in $t_{\max i}$, the maximum the person is prepared to pay for his insurance premium. Other models are possible, however, as will be shown in Section 6.

4.2. Lottery ticket

Arguments similar to the insurance example may be applied to the case of a lottery. The assumption is made that while peripheral considerations, such as advertising for the lottery and the design and colour of the ticket, may act as inducements to take part, they will have no significant effect on the person's decision on what should be his maximum acceptable ticket price. It is then reasonable to expect that the risk-aversion, $\varepsilon_{t_{\max i}}$, for person i that is associated with his maximum acceptable ticket price, $t_{\max i}$, will depend on and only on:

- z , the value of the prize.
- w_i , the person's initial wealth.
- p , the probability of winning.
- $t_{\max i}$, the maximum the person is prepared to pay for his ticket.
- $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, the individual's personal traits relevant to taking decisions under risk.

The mathematics now follow a path fully analogous to that of Section 4.1, so that

$$\varepsilon_{t_{\max i}} = g\left(\frac{z}{t_{\max i}}, \frac{w_i}{t_{\max i}}, p, \tau_{1i}, \tau_{2i}, \dots, \tau_{mi}\right) \quad (37)$$

Once again each of the three dimensionless numbers, $w_i/t_{\max i}$, $z/t_{\max i}$ and p , is constant during the process of comparing outturns, and, since the personal traits, $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, remain constant over long periods, it follows from Eq. (38) that $\varepsilon_{t_{\max i}}$ must be constant also.

In an analogy with the insurance case, Eq. (37) cannot confirm Eq. (18), but the two equations agree on the same three independent variables particular to the decision in hand. Personal traits, $\tau_{1i}, \tau_{2i}, \dots, \tau_{mi}$, are ignored in Eq. (18).

Similar arguments to those used in Section 4.1 allow Eq. (37) to be extended to a test maximum premium, $t_{\max i}^{(test)}$, and associated risk-aversion, $\varepsilon_{t_{\max i}^{(test)}}$.

5. The utility function produced when risk-aversion stays constant at the point of decision

By the model of [1] discussed above, risk-aversion will stay the same during the process of comparing outturns, so that we may now proceed to the integration of Eq. (5) under the condition that $\varepsilon = \text{constant}$. It may first be noted that

$$\frac{d}{dw} \ln u'(w) = \frac{u''(w)}{u'(w)} = -\frac{\varepsilon}{w} \quad (38)$$

Hence:

$$\ln u'(w) = -\int \frac{\varepsilon}{w} dw \quad (39)$$

so that $u'(w) = \exp\left(-\int \frac{\varepsilon}{w} dw\right)$ and

$$u(w) = \int e^{-\int \frac{\varepsilon}{w} dw} dw \quad (40)$$

Given that $\varepsilon = \text{constant}$, then

$$-\int \frac{\varepsilon}{w} dw = -\varepsilon \int \frac{1}{w} dw = -\varepsilon \ln w + \alpha = \ln w^{-\varepsilon} + \alpha \quad (41)$$

where $\alpha = \text{constant}$. It follows further that

$$\begin{aligned} u(w) &= \int e^{\ln w^{-\varepsilon} + \alpha} dw = \int e^{\ln w^{-\varepsilon}} e^{\alpha} dw \\ &= e^{\alpha} \int w^{-\varepsilon} dw = \begin{cases} e^{\alpha} \frac{w^{1-\varepsilon}}{1-\varepsilon} + \beta & \text{for } \varepsilon \neq 1 \\ e^{\alpha} \ln w + \gamma & \text{for } \varepsilon = 1 \end{cases} \end{aligned} \quad (42)$$

where β and γ are constants and e^{α} is a positive constant. Hence, using the result of von Neumann and Morgenstern that utility functions may be regarded as valid to a positive linear transformation, we may remove the constant terms, β and γ , and e^{α} , to produce:

$$u(w) = \begin{cases} \frac{w^{1-\varepsilon}}{1-\varepsilon} & \text{for } \varepsilon \neq 1 \\ \ln w & \text{for } \varepsilon = 1 \end{cases} \quad (43)$$

Further progress may be made by noting that

$$\ln x = \ln x - \ln 1 \quad (44)$$

implying that the lower term in Eq. (43) may be regarded as the difference between the utility of wealth, w , and the utility of one unit of money. Transferring the same datum to the other term in Eq. (43) gives Eq. (6), repeated below:

$$u(w) = \begin{cases} \frac{w^{1-\varepsilon} - 1}{1-\varepsilon} & \text{for } \varepsilon \neq 1 \\ \ln w & \text{for } \varepsilon = 1 \end{cases} \quad (6)$$

Eq. (6) may be regarded as defining a utility function in its own right, since the subtraction of the constant, $1/(1-\varepsilon)$, from the top line of the utility function of Eq. (43) will produce another utility function under the von Neumann and Morgenstern rules. Eq. (6) may also be regarded as the difference between the utility of wealth and the utility of one unit of wealth, when both are calculated from Eq. (43). Following the precedent of Cowell and Gardiner [26], who attributed the utility function of Eq. (6) to Atkinson [27], it is given the name the Atkinson utility function in this paper, although its basic formulation was obviously apparent earlier to Pratt [16].

The simplest Power utility may be found by multiplying the right-hand side of the top line of Eq. (43) by $1-\varepsilon$ for the case $\varepsilon < 1$ to give:

$$u(w) = w^{1-\varepsilon} \text{ for } \varepsilon < 1 \quad (45)$$

Like the more complicated Atkinson utility function, the Power utility may be regarded either as a utility function

or as the difference between two utility functions, in this case between the utility function of Eq. (45) applied to wealth and the same utility function applied to zero wealth. The Power utility function is slightly easier to apply, but the Atkinson utility function has the advantage that it caters for situations where the risk-aversion is greater than unity, offering a smooth transition as ε increases from below 1.0, through unity and then above 1.0, as illustrated in [1].

6. Measuring risk-aversion from the maximum acceptable insurance premium using a utility function from the Power family

A variable important for decision taking, namely the reluctance to invest, has been defined [29,20] as the difference in utility before and after investment, normalised to the starting utility relative to the utility of either no money (for the basic Power utility) or one unit of money for the Atkinson utility function. When the Power utility function is used, the symbol, R_{120P} , is employed for the reluctance to invest:

$$\begin{aligned} R_{120P} &= \frac{(1-p)u(w) + pu(w-z) - u(w-t)}{u(w)} \\ &= \frac{(1-p)w^{1-\varepsilon} + p(w-z)^{1-\varepsilon} - (w-t)^{1-\varepsilon}}{w^{1-\varepsilon}} \\ &= 1 - p + p(1-z/w)^{1-\varepsilon} - (1-t/w)^{1-\varepsilon} \\ &= 1 - p + p(1-c)^{1-\varepsilon} - (1-b)^{1-\varepsilon} \quad \text{for } \varepsilon \neq 1 \end{aligned} \quad (46)$$

where b is the normalised premium: $b = t/w$, while c is the normalised potential loss: $c = z/w$. (The subscript, i , has been dropped in this development simply in order to reduce the 'busy' nature of the equations. The results are still particular to the individual or organisation.) Meanwhile the reluctance to invest for the Atkinson utility function when $\varepsilon \neq 1$ is:

$$\begin{aligned} R_{120A} &= \frac{(1-p)u(w) + pu(w-z) - u(w-t)}{u(w)} \\ &= \frac{(1-p)(w^{1-\varepsilon} - 1) + p((w-z)^{1-\varepsilon} - 1) - ((w-t)^{1-\varepsilon} - 1)}{w^{1-\varepsilon} - 1} \\ &= \frac{w^{1-\varepsilon}}{w^{1-\varepsilon} - 1} \left(1 - p + p(1-c)^{1-\varepsilon} - (1-b)^{1-\varepsilon} \right) \quad \varepsilon \neq 1 \end{aligned} \quad (47)$$

so that

$$R_{120A} = \frac{w^{1-\varepsilon}}{w^{1-\varepsilon} - 1} R_{120P} \quad \varepsilon \neq 1 \quad (48)$$

In the case that $\varepsilon = 1$, the top line of Eq. (47) may be developed as

$$\begin{aligned} R_{120A} &= \frac{(1-p) \ln w + p \ln(w-z) - \ln(w-t)}{\ln w} \\ &= \frac{p \ln(1-z/w) - \ln(1-t/w)}{\ln w} \\ &= \frac{p \ln(1-c) - \ln(1-b)}{\ln w} \quad \varepsilon = 1 \end{aligned} \quad (49)$$

For any given premium that a subject is prepared to pay, it is required to find the value of risk-aversion, ε , that minimises the reluctance to invest, which is equivalent to maximising the desire to invest. This value of risk-aversion is known as the individual's 'permission point', ε_{ppi} , since it is the value at which the decision maker will feel most content with his decision to pay the specified premium [29]. Differentiating Eq. (48) gives the rate of change of R_{120A} with ε as:

$$\frac{dR_{120A}}{d\varepsilon} = \frac{w^{1-\varepsilon}}{w^{1-\varepsilon} - 1} R'_{120P} + R_{120P} \frac{w^{1-\varepsilon}}{(w^{1-\varepsilon} - 1)^2} \ln w \quad (50)$$

and the valley minimum of R_{120A} with respect to ε is thus given by:

$$(w^{1-\varepsilon} - 1) R'_{120P} - R_{120P} \ln w = 0 \quad (51)$$

where $R'_{120P} = dR_{120P}/d\varepsilon$ is given by:

$$R'_{120P} = p(1-c)^{1-\varepsilon} \ln(1-c) - (1-b)^{1-\varepsilon} \ln(1-b) \quad (52)$$

To illustrate the concepts, let us examine the case where insurance cover is being considered against a possible small loss. The individual is assumed to have wealth of £1000 and wishes to insure a washing machine he has just bought for £250 against possible complete breakdown and replacement, an event that occurs with a probability of 0.1, so that the expected loss is £25. Fig. 1 gives shows the behaviour of the reluctance to invest against risk-aversion for a person prepared to pay a premium of £25, the same as the expected loss. Clearly the individual would be happy (just) to pay this premium if his risk-aversion were $\varepsilon_{t \max i} = 0$, the value of risk-aversion at which the locus of the line cuts the horizontal axis, but his utility would be maximised at the minimum of the risk-aversion locus, when $\varepsilon_i = 0.55$, the 'permission point', ε_{ppi} [20].

The permission point in this instance coincides with a reluctance to invest of -8.53×10^{-4} , the negative value indicating a desire to invest in the insurance premium. However, the person being forced to his limit on what he would be prepared to pay for insurance will be prepared to increase his permission point until it reaches the 'point of indiscriminate decision', $\varepsilon_{indis-i}$. Now the absolute value of his reluctance to invest will have fallen to the limiting level at which he is still able to discriminate between out-turn utilities: $|R_{120Ai}|_{\varepsilon_i = \varepsilon_{ppi}} = R_{120Ai}^{(lim)}$. Any increase in risk-aversion will now lead to a random decision on whether to invest or not, so no decision should be taken when $\varepsilon_i > \varepsilon_{indis-i}$.

Thus the same data on insurance premia allow the individual's limiting reluctance to invest, $R_{120Ai}^{(lim)}$, to be estimated in addition to the permission point, ε_{ppi} . Fig. 2 shows the locus of reluctance to invest versus risk-aversion at insurance premia of £26, £28 and £30. As his MAP increases, the individual's risk-aversion needed to sanction that MAP also increases, but this is accompanied by a decrease in the absolute value of the reluctance to invest at which he can still discriminate. Thus if the individual's MAP is found to be £26, then his permission point will be $\varepsilon_{ppi} = 0.74$ and his limiting reluctance to invest will be

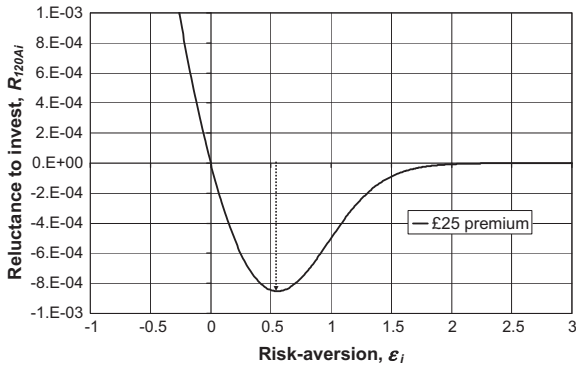


Fig. 1. Insurance case of Section 6: the behaviour of reluctance to invest, R_{120Ai} , showing the minimum value of risk-aversion ($\varepsilon_i = 0.0$) at which a person offered an insurance premium at £25 could sanction the decision and the actual value at which the decision to invest would be made ($\varepsilon_i = 0.55$).

$R_{120Ai}^{(\text{lim})} = 4.57 \times 10^{-4}$. If his MAP is £28, then $\varepsilon_{ppi} = 1.15$ and $R_{120Ai}^{(\text{lim})} = 7.74 \times 10^{-5}$, while if his MAP is £30, then $\varepsilon_{ppi} = 1.63$ and $R_{120Ai}^{(\text{lim})} = 6.47 \times 10^{-6}$.

The figures for the permission point (the actual risk-aversion for taking the decision to buy insurance) may be compared with those produced by the Taylor series model, based on an assumption of utility break-even, equivalent to a reluctance to invest of zero: $R_{120Ai} = 0$. Thus for a MAP of $t_{\text{max}i} = £26$, $\varepsilon_{BEi} = 0.36$, for $t_{\text{max}i} = £28$, $\varepsilon_{BEi} = 1.1$, while for $t_{\text{max}i} = £30$, $\varepsilon_{BEi} = 1.87$, a different set of risk-aversions from those cited in the previous paragraph. Fig. 3 displays, for a wide range of possible MAPs:

- the permission point, ε_{ppi} (the true value of risk-aversion at which a decision to buy insurance will be taken),
- the utility break-even value of risk-aversion, ε_{BEi} , predicted by the Atkinson utility function,
- the utility break-even value of risk-aversion, ε_{BEi} , predicted by a Taylor series expansion.

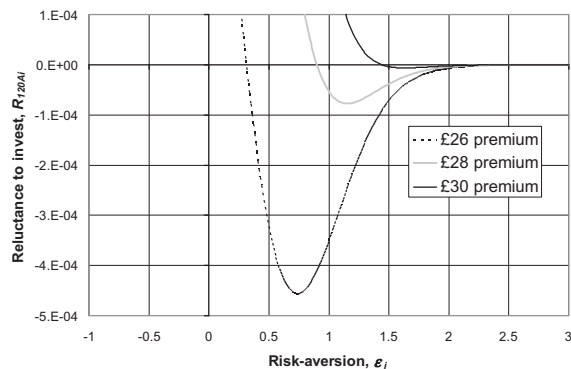


Fig. 2. Insurance case of Section 6: reluctance to invest versus risk-aversion for insurance premia of £26, £28 and £30.

The Taylor series figure for ε_{BEi} may be seen to come close to the value of ε_{BEi} predicted by the Atkinson utility function for MAPs between £21 and £26, but it diverges from the accurate, utility-function-generated figure outside this range. Of course, both of these figures are approximations only to the actual figure needed, which is the permission point, ε_{ppi} . There is evidence for the utility function figure for ε_{BEi} converging towards ε_{ppi} at high values of MAP, but the Taylor series value of ε_{BEi} diverges.

Fig. 4 shows the limiting reluctance to invest over a range of possible MAPs for insurance premium. Premia of £31 and above lead to $R_{120Ai}^{(\text{lim})} < 5 \times 10^{-6}$, a level likely to be beyond everyone's ability to discriminate between outcomes. This restricts the upper bound for possible MAPs to a rather low level. This limit, only £6 (~20%) up on the expected loss is thus very tight for this example where rather small sums of money are at stake, indicating that results derived for this case may be sensitive to quantisation noise on the MAP recorded.

To complete the picture, Fig. 5 shows the behaviour of an individual's reluctance to invest, R_{120Ai} , versus his risk-aversion, ε_i , for very low insurance premia. These premia are well below the expected loss (£25), and it is therefore unlikely that they would be offered by any insurance company except possibly as a loss-leader. The permission point, ε_{ppi} , has moved into the left-hand plane and become negative. As the insurance premium becomes cheaper, so the permission point, ε_{ppi} , becomes ever more negative.

The theoretical possibility of highly negative risk-aversions raises the question as to whether everyone is equipped to feel that degree of risk confidence. The possibility that there may be a variation in the highest risk confidence (=most negative risk-aversion) that different people are able to feel may be modelled by assigning each person a lower level of risk-aversion, $\varepsilon_i^{(\text{low})}$, below which his risk-aversion cannot be reduced. The interest of highly negative risk-aversions subject to a limit, $\varepsilon_i^{(\text{low})}$, is mainly theoretical in the insurance case, with $\varepsilon_i^{(\text{low})}$ being included for completeness only. However, it has more direct implications for the case of a lottery.

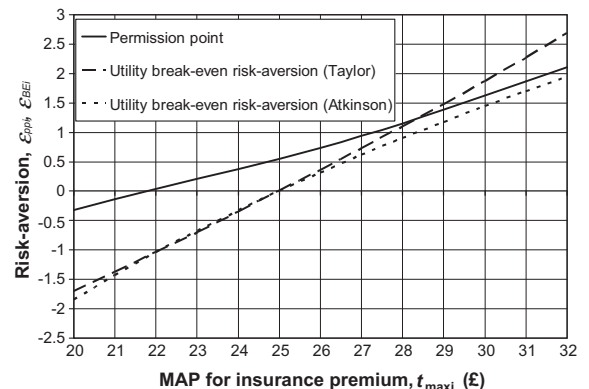


Fig. 3. Insurance case of Section 6: permission point, ε_{ppi} , minimum risk-aversion, ε_{mini} found from Taylor series expansion and from the utility function, for a range of possible insurance MAPs.

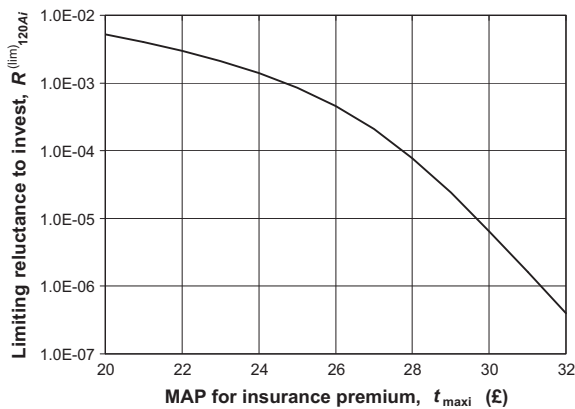


Fig. 4. Insurance case of Section 6: absolute value of the limiting reluctance to invest for a range of possible MAPs for insurance.

Summarising: the location of the minimum reluctance to invest, $(\varepsilon_{ppt}, R_{120Ai}^{(lim)})$, in the plane of reluctance to invest versus risk-aversion allows inferential measurements to be made of both the risk-aversion, $\varepsilon_{t_{maxi}} = \varepsilon_{ppt}$, associated with the person's maximum acceptable premium, t_{maxi} , and his limiting reluctance to invest, $R_{120Ai}^{(lim)}$. From an alternative viewpoint, it is possible to find $\varepsilon_{t_{maxi}}$ if the variables particular to the current decision are known: z , w_i , p and t_{maxi} , together with the individual's limiting reluctance to invest, $R_{120Ai}^{(lim)}$. Including his lower level of risk-aversion, $\varepsilon_i^{(low)}$, for completeness, Eq. (36) may be expanded to give the general result that the person's risk-aversion at his maximum acceptable premium is a function of up to 5 dimensionless variables:

$$\varepsilon_{t_{maxi}} = g\left(\frac{z}{t_{maxi}}, \frac{w_i}{t_{maxi}}, p, R_{120Ai}^{(lim)}, \varepsilon_i^{(low)}\right) \quad (53)$$

Thus the number of relevant personal traits, m , emerges as two in the insurance case: $\tau_{1i} = R_{120Ai}^{(lim)}$ and $\varepsilon_i^{(low)}$, both of which are dimensionless, as foreshadowed in Section 4.

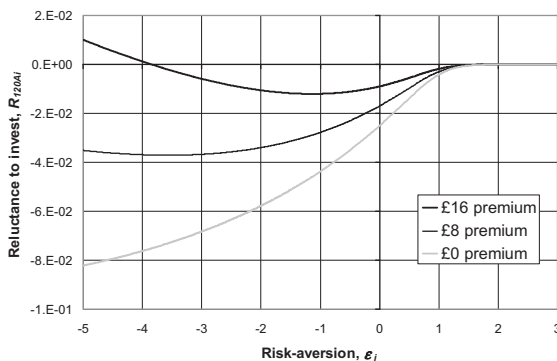


Fig. 5. Insurance case of Section 6: reluctance to invest versus risk-aversion at very low insurance premia.

7. Measuring risk-aversion from insurance premia where the possible loss is a sizeable fraction of the individual's wealth

An enhanced perspective on risk-aversion is given in the second case considered, that of a typical UK adult, assumed to possess £180,000 of wealth, mostly contained in his house, for which he is interested in taking out buildings insurance. The potential loss is £150,000, with probability 0.001, implying an expected loss of £150. Fig. 6 compares the results for the buildings insurance case. It is clear from the Figure that the Taylor series expansion method does not give an accurate indication even of the minimum risk-aversion except for a very restricted region. This is not unexpected, in view of the fact that the potential loss is a large fraction of the subject's wealth, thus contravening an assumption of the Taylor series expansion.

The capability of the utility function method to find the risk-aversion in more general, realistic situations brings with it two immediate advantages. First, the absolute value of the reluctance to invest does not fall to 5×10^{-6} or below, the level likely to be beyond everyone's ability to discriminate between outcomes, until the insurance premium has risen to about £350, which is more than twice the expected loss. See Fig. 7. This allows the measurement, potentially with good accuracy, of a fairly large range of insurance premia with which different people might feel happy. Moreover, the change in risk-aversion with small changes in acceptable premium is much less in the house insurance case than in the case of washing machine insurance, making the results less sensitive to quantisation noise.

8. Measuring risk-aversion from the maximum acceptable price for a lottery ticket using a Power-family utility function

In the case of the lottery, the reluctance to invest under a Power utility function, R_{120P} , is:

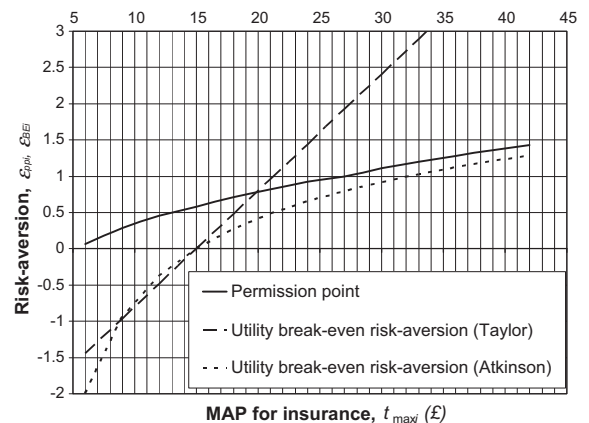


Fig. 6. Insurance case of Section 7: comparison between the utility function and Taylor series expansion methods.

$$\begin{aligned}
 R_{120P} &= \frac{u(w) - (1-p)u(w-t) - pu(w-t+z)}{u(w)} \\
 &= \frac{w^{1-\varepsilon} - (1-p)(w-t)^{1-\varepsilon} - p(w-t+z)^{1-\varepsilon}}{w^{1-\varepsilon}} \\
 &= 1 - (1-p)(1-t/w)^{1-\varepsilon} - p(1-t/w+z/w)^{1-\varepsilon} \\
 &= 1 - (1-p)(1-b)^{1-\varepsilon} - p(1-b+c)^{1-\varepsilon} \\
 &\quad \text{for } \varepsilon \neq 1
 \end{aligned} \quad (54)$$

where b is the normalised price of the lottery ticket: $b = t/w$, while c is the normalised potential win: $c = z/w$. (Again the subscript, i , has been omitted in this development simply in order to reduce the ‘busy’ nature of the equations. The results are still particular to the individual or organisation.) Meanwhile the reluctance to invest found from using the Atkinson utility function when $\varepsilon \neq 1$ is:

$$\begin{aligned}
 R_{120A} &= \frac{u(w) - (1-p)u(w-t) - pu(w-t+z)}{u(w)} \\
 &= \frac{w^{1-\varepsilon} - 1 - (1-p)((w-t)^{1-\varepsilon} - 1) - p((w-t+z)^{1-\varepsilon} - 1)}{w^{1-\varepsilon} - 1} \\
 &= \frac{w^{1-\varepsilon}}{w^{1-\varepsilon} - 1} \left(1 - (1-p)(1-b)^{1-\varepsilon} - p(1-b+c)^{1-\varepsilon} \right) \quad \varepsilon \neq 1
 \end{aligned} \quad (55)$$

so that, as with the insurance case, we may write

$$R_{120A} = \frac{w^{1-\varepsilon}}{w^{1-\varepsilon} - 1} R_{120P} \quad (56)$$

Expanding the top line of Eq. (55) for the case of $\varepsilon = 1$ gives

$$\begin{aligned}
 R_{120A} &= \frac{\ln w - (1-p)\ln(w-t) - p\ln(w-t+z)}{\ln w} \\
 &= \frac{\ln w - (1-p)(\ln w + \ln(1-t/w)) - p(\ln w + \ln(1-t/w+z/w))}{\ln w} \\
 &= \frac{-(1-p)\ln(1-b) - p\ln(1-b+c)}{\ln w} \quad \varepsilon = 1
 \end{aligned} \quad (57)$$

or

$$R_{120A} = \frac{p(\ln(1-b) - \ln(1-b+c)) - \ln(1-b)}{\ln w} \quad \varepsilon = 1 \quad (58)$$

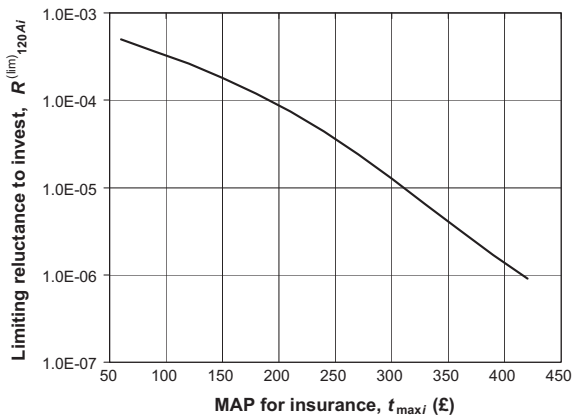


Fig. 7. Insurance case of Section 7: limiting reluctance to invest over a range of MAPs.

Eq. (51) may be used again to find the extremum of the curve of R_{120A} against ε , but in this, lottery case, the extremum found will be a peak. In this case, R_{120P} is given by Eq. (54) and differentiating that equation with respect to risk-aversion gives:

$$\begin{aligned}
 R'_{120P} &= (1-p)(1-b)^{1-\varepsilon} \ln(1-b) \\
 &\quad + p(1-b+c)^{1-\varepsilon} \ln(1-b+c)
 \end{aligned} \quad (59)$$

Consider the example of a person with wealth, w , of £1000 wondering whether or not to buy a ticket in a lottery with a prize, z , of £10,000, so that $c = z/w = 10$. Let us suppose that the probability of winning the prize, p , is 1 in 1000 and so the expected value of the lottery is £10. Suppose that the price of a lottery is gradually raised from £1 in a procedure analogous to an auction, and that one person, let us call him person 1, stops bidding at £4.63, signifying that this is his maximum acceptable price. Fig. 8 shows his reluctance to invest against his risk-aversion for this MAP. Assuming the person is being pushed to his most adventurous in deciding on his MAP, t_{maxi} , his corresponding risk-aversion, $\varepsilon_{t_{maxi}}$, will have been driven down to his limiting value of risk-aversion, $\varepsilon_i^{(low)}$. Since he is still prepared to buy a ticket at this price, his reluctance to invest cannot be positive, and, since he has reached the limit of what he will pay, his reluctance to invest will be zero. The corresponding value of risk-aversion is $\varepsilon_{t_{maxi}} = 0.5 = \varepsilon_i^{(low)}$.

This may be seen more clearly by considering Fig. 9, which shows the reluctance to invest versus risk-aversion for ticket prices of £4.49, £4.63 and £4.77. Anyone who is prepared to pay £4.49 for a ticket must have a lower threshold on risk-aversion of 0.52 or below; anyone prepared to pay £4.63 for the same ticket must have a lower threshold of 0.50 or below, while the lower threshold of someone still in the auction at £4.77 must have a lower threshold of 0.48 or below. Person 1 exited the auction after the price reached £4.49 but before it rose to £4.77, and so his lower threshold on risk-aversion must be below 0.52 but cannot be 0.48 or lower. Hence $\varepsilon_1^{(low)}$ must lie between 0.48 and 0.52: $0.48 < \varepsilon_1^{(low)} < 0.52$. Assuming the auction price is raised continuously, the range can be tightened, allowing us to deduce that someone leaving the

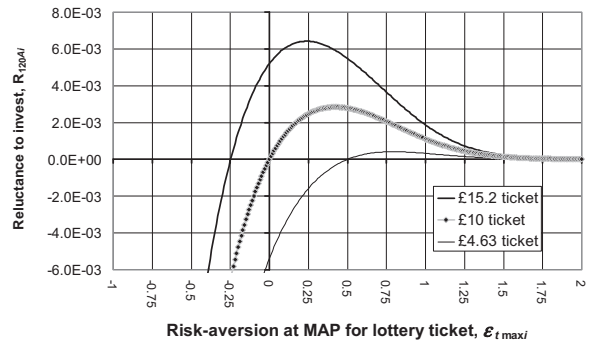


Fig. 8. Lottery case of Section 8: reluctance to invest versus risk-aversion for a range of t_{maxi} .

auction immediately after offering £4.63 must have a lower threshold on risk aversion of $\varepsilon_i^{(low)} = 0.5$.

While $\varepsilon_{t\max i} = 0.5$ will almost certainly be the dominant mode, it needs to be borne in mind that another mode of behaviour is conceivable, should the individual have a very high value of limiting risk-aversion, $\varepsilon_i^{(low)}$. For example, if the lower threshold on risk-aversion of person 1 were 1.61 or more, viz. $\varepsilon_i^{(low)} \geq 1.61$, while, at the same time, his limiting value of reluctance to invest was $R_{120A1}^{(lim)} = 3 \times 10^{-5}$, he might find himself caught in a region where he could not discriminate between the advantages of buying a ticket at £4.63 or not. Were he able to become rather more risk-confident and test his reaction to a risk-aversion of 1.0, for example, he would decide against buying the ticket, but this strategy is ruled out for him by his high value of $\varepsilon_i^{(low)}$. Hence his decision would become indeterminate, and it is possible that he might buy at ticket at £4.63 even though he was experiencing a very high risk-aversion. This sort of dualism associated with risk decisions was observed by Atkinson [30] in the behaviour of children at play. The phenomenon has been explained previously, [17], in terms of risk-aversion.

Fig. 8 shows also cases where the MAP is £10 and £15.2, from which the most likely risk-aversions, $\varepsilon_{t\max i}$, are 0.0 and -0.25, although it would still be possible theoretically for someone experiencing very high levels of risk-aversion to purchase these tickets, if his value of $\varepsilon_i^{(low)}$ were sufficiently high.

The application of an Atkinson utility function is able to give a further important insight into the behaviour of inveterate gamblers. For it is possible for someone who has a very low limit for risk-aversion, $\varepsilon_i^{(low)} \ll 0$, to regard even very expensive lottery tickets as worth buying, even at many times their expected return. Fig. 10 shows that people with very low limiting risk-aversions, $\varepsilon_i^{(low)} = -1.0$, -1.2 and -1.4, could be content to buy tickets for £60, £90 and £150 respectively. Despite the fact that the expected loss is significant in all these cases, someone with a very low limit on his risk-aversion could see a positive gain in expected utility.

The mechanism displayed in Fig. 10 provides a rational, economic explanation for why some gamblers take huge

risks, even though they know that they are almost certain to lose. Because the low limit on his risk-aversion is set too low, a problem gambler can experience a highly negative risk-aversion, enabling him to feel a highly negative reluctance to invest in a very risky venture, equivalent to a high desire to invest.

9. Discussion

The economic parameter, risk-aversion, has profound importance in economics, but attempts to measure it have been sporadic only. Indeed, some respected actuarial scientists believe it is impossible to determine the form of a utility function and hence the decision-maker's risk-aversion in practice [31]:

"It is impossible to determine which utility functions are used 'in practice'. Utility theory merely states the existence of a utility function. We could try to reconstruct a decision maker's utility function from the decisions he takes, by confronting him with a large number of questions like: 'Which premium P are you willing to pay to avoid a loss ... that could occur with probability q ?' ... In practice, we would soon experience the limitations of utility theory: the decision maker will grow increasingly irritated as the interrogation continues and his decisions will become inconsistent".

Many economists, too, regard utility as a useful theoretical construct only, and not one that is open to measurement, a position described as the 'standard model' of economics in Chapter 1 of [32]:

"Utility maximisation by individuals sits nicely with utilitarianism in general, or with other kindred approaches, such as the maximisation of an equity-prefering social welfare function that respects individual choices. In this framework, there is no need to measure utility, because we have everything that we need. People's choices reveal everything about their preferences that we need to know. Of course, there would be no harm in measuring utility, if we could do so. But it is unnecessary and, indeed, doing away with utility, marginal utility, and interpersonal comparisons of utility was long thought to be one of the great achievements of modern economics."

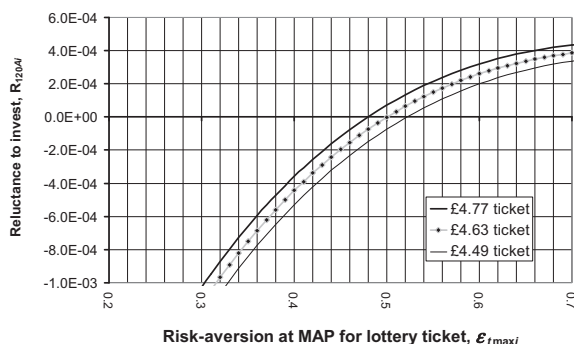


Fig. 9. Lottery case of Section 8: reluctance to invest versus risk-aversion for $\varepsilon_{t\max i} = £4.49, £4.63$ and $£4.77$.

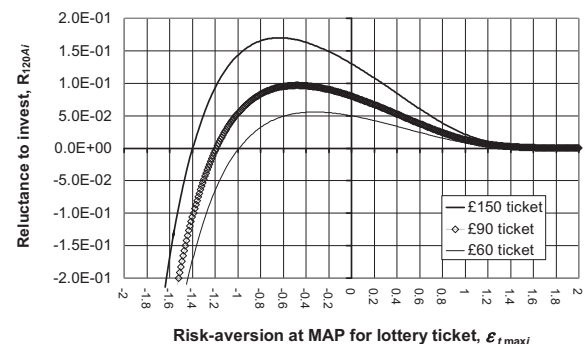


Fig. 10. Lottery case of Section 8: reluctance to invest versus risk-aversion for very expensive lottery tickets.

The view espoused by Hicks [7] has so far won over more economists than that advanced by von Neumann and Morgenstern [8].

And it must be conceded that attempting to measure risk-aversion using traditional approaches based on Taylor series expansions are likely to be problematic because of the requirement for small excursions from the starting wealth. This restriction applies not only to the lottery ticket price and the insurance premium, where it may not cause too many problems, but also to the lottery prize and the potential loss, where its imposition will certainly be awkward to accommodate.

The constraint introduced by the Taylor series that the prize from the lottery must not be comparable with the starting wealth of any of the individuals means that the test gamble must be rendered of artificially low importance to all the individuals in the sample. The result of this is that it will be rational for all of them to employ a low risk-aversion in their assessment of it. Any study trying to pick out differences between groups will thus be rendered problematic, as reported in [28], where it was found hard to differentiate between the risk-aversions of entrepreneurs and others on the basis of a lottery with prize set at only 1000 guilders (about €450).

Moreover, a logical gap has been found in the conventional, Taylor series method of analysing the risk-aversion experienced in buying a lottery ticket at the individual's maximum acceptable price. Taking the risk-aversion at utility break-even as the value used by the person in deciding his maximum ticket price requires an unacknowledged further mechanism. It is necessary to posit that the individual has a pre-set lower threshold, $\varepsilon_i^{(low)}$ below which individual i may not reduce his risk-aversion.

The restriction to small deviations inherent in Taylor series analysis presents a problem in using the technique to infer risk-aversion from the maximum acceptable insurance premium. The technique is valid only for potential losses that are relatively unthreatening to the individuals' wealth, where it is rational for all the individuals concerned to employ a low risk-aversion. So while insurance of domestic devices of relatively low value is open to analysis using a Taylor series, bigger potential losses such as house insurance are not. Since it is only when the loss is a significant fraction of his wealth that a rational person is predicted to exhibit a high risk-aversion [20,21], a good signal-to-noise ratio is to be expected only in the latter case. Behaviour in a more demanding insurance situation would need to be examined in any study seeking to establish whether different risk-aversions were exhibited by different types of individuals.

A further weakness in the conventional application of the Taylor series expansion to determine risk-aversion from insurance data consists in the fact that it is generally incorrect to assume that the risk-aversion at utility break-even will be the value experienced by the individual when paying his maximum acceptable premium for insurance. The correct value is the permission point, ε_{ppi} , when it coincides with the point of indiscriminate decision: $\varepsilon_{ppi} = \varepsilon_{indis-i}$. This coincidence of values will occur when the reluctance to invest, R_{120Ai} , has reached such a low

level, $R_{120Ai}^{(lim)}$ that the individual can no longer discriminate between the expected utilities of the two courses of action under consideration.

The lower threshold on risk-aversion, $\varepsilon_i^{(low)}$ and the limiting reluctance to invest, $R_{120Ai}^{(lim)}$, are two further dimensionless parameters that are particular to the individual and should be considered for measurement in addition to risk-aversion.

The lottery and insurance cases are particularly useful to consider because they are canonical representations of more complex situations where decisions under uncertainty are required. However an insight into problem gambling is provided by the finding that a lower threshold on risk-aversion, $\varepsilon_i^{(low)}$, particular to the individual, is one of the traits necessary to explain the maximum acceptable price for a lottery ticket. The model suggests that the lower threshold on risk-aversion, $\varepsilon_i^{(low)}$, is what differentiates problem gamblers from the majority of the population. The parameter is set too low in gamblers making consistent losses.

10. Measuring risk-aversion: the challenge

Information of great economic value is potentially available from exercises to measure people's risk-aversions when faced with different decisions. A theoretical framework has now been developed to allow an accurate measurement of an individual's risk-aversion from his responses both to a lottery and to insurance.

There are still difficulties to overcome. The first lies in devising a process to elicit from the individual his maximum acceptable price for a lottery ticket or his maximum acceptable insurance premium, and ensuring that his supposedly highest value is truly representative of what he would offer in a real situation. Then there is the fundamental problem that the individual will exhibit multiple values of risk-aversion depending on the importance of the decision to him.

Nevertheless the use of a single class of utility function with risk-aversion as sole parameter, as recommended in this paper, brings with it both clarification and simplification, enabling a more reliable measurement of risk-aversion to be made.

This comes at a time when there is considerable interest in economic indicators that reflect the subjective feelings of the person taking the decision [32]. Clearly risk-aversion is one such indicator rooted in people's feelings. Moreover, it has the advantage that its definition in mathematical terms allows a rigorous, inferential measurement to be made. This is also the case for the related measurements of the lower threshold on risk aversion, $\varepsilon_i^{(low)}$, and the limiting reluctance to invest, $R_{120Ai}^{(lim)}$.

Measurement specialists with an interest in 'soft' measurement or behavioural economists might care to take up the challenge. Alternatively, the data held within the insurance industry might make an insurance company the best candidate to carry out an exercise that might bring them and their clients useful commercial benefits.

11. Conclusions

The paper has reviewed the historical development of the concepts of utility and risk-aversion, starting nearly 300 years ago. It is a tribute to the distinguished pioneers that much of their thinking retains a relevance today. Risk-aversion was identified as an important dimensionless parameter in 1964, but has tended to be regarded since that time as a mathematical construct. However the argument has been made in this paper that risk-aversion is the fundamental descriptor of the feeling guiding the person taking a decision between two alternatives when the outcome of one or both is uncertain. The fact that risk-aversion is dimensionless (unlike the ‘coefficient of absolute risk aversion’, a name that seems to claim too much) gives it the capability to model situations where what is at risk can be a variety of different things of importance to the individual, not just money, but status, happiness and so on.

Conventionally, inferential measurements of risk-aversion have been made by expanding about the utility of starting wealth using a Taylor series. However, this approach can be valid only when the stakes are low. In such circumstances one would expect risk-aversion to be close to zero for everyone, making its measurement difficult and subject to quantisation noise. It becomes difficult then to distinguish any possible difference in risk-aversion between two people of different character, e.g. an entrepreneur and a salaried professional.

The conventional, Taylor series approach appears to have been motivated in part, at least, by a desire to render the results independent of the precise form of the utility function. However, taking the case of decisions about money, if risk-aversion is regarded as a function of the present state of wealth, rather than as being altered by an act of imagination in response to possible future changes to that wealth, the valid classes of utility function reduce to one. This is the Power family, for which risk-aversion is the sole parameter. The grounding of utility in a single class of function allows more direct analysis of decisions.

This more direct analysis has shown up two further shortcomings associated with the use of the conventional, Taylor series expansion. In the case of the lottery, it has been shown that assigning a unique value to the risk-aversion based on the maximum price that someone is prepared to offer for a lottery ticket requires a further mechanism not normally mentioned, namely a lower threshold for risk-aversion unique to the person. In the case of insurance, the conventional thinking is that the decision to invest in insurance is taken at utility break-even, when the reluctance to invest is zero: $R_{120Ai} = 0$, whereas this paper has suggested that the decision is actually taken at the minimum of reluctance to invest, when $dR_{120Ai}/d\varepsilon = 0$.

Dimensional analysis has been used to show that the risk-aversion when deciding on the maximum acceptable price for a lottery ticket or on the maximum acceptable insurance premium can be represented as a function of three dimensionless variables particular to the decision at hand and a set of dimensionless character traits. The

three dimensionless variables are: (i) the prize or loss denominated in terms of the maximum acceptable price for the lottery ticket or the insurance premium, (ii) the individual's or organisation's wealth, denominated in the same units and (iii) the probability of winning the prize or making the loss. Risk-aversions derived using Taylor series methods are found to be functions of these 3 dimensionless variables, with no further allowance being made for character traits. Risk-aversions derived using the Atkinson utility function depend on the same 3 dimensionless variables but also on two new, nondimensional character traits: the limiting reluctance to invest, $R_{120Ai}^{(lim)}$, and the lower threshold on risk-aversion, $\varepsilon_i^{(low)}$. Inferential measurements of these parameters should be possible and fruitful. The latter parameter, in particular, may offer insights into problem gambling.

The paper has established a theoretical framework for the measurement of risk-aversion and the two related parameters, limiting reluctance to invest and lower threshold on risk-aversion. Successful measurement offers the chance for evidence-based utility calculations to be made, with the potential for new economic insights as a result.

Acknowledgements

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Appendix A. The range of lottery ticket price, t , over which $\varepsilon(\max|t, w_i)$ is a decreasing function of t

Consider Eq. (17) repeated below:

$$\varepsilon(\max|t, w_i) = \frac{2w_i(pz - t)}{pz^2 - 2pzt + t^2} \quad (17)$$

The properties of the equation mean that risk-aversion, ε , will be a decreasing function of t for $0 < t < t_0(1 + \sqrt{o_A})$, where $t_0 = pz$ is the expected value of the lottery ticket, equivalent to the utility of the ticket for someone with a neutral risk-aversion, namely $\varepsilon = 0$, while o_A is the odds against winning: $o_A = (1 - p)/p$.

The upper point of the range will occur when the differential, $d\varepsilon/dt$, ceases being negative, where:

$$\begin{aligned}\frac{d\varepsilon}{dt} &= \frac{-2w_i(pz^2 - 2pzt + t^2) - 2w(pz - t)(-2pz + 2t)}{(pz^2 - 2pzt + t^2)^2} \\ &= \frac{2w_i(t^2 - 2pzt + 2p^2z^2 - pz^2)}{(pz^2 - 2pzt + t^2)^2}\end{aligned}\quad (\text{A.2})$$

Risk-aversion, ε , will decrease to a minimum at the point where $d\varepsilon/dt = 0$, corresponding to the positive root of $t^2 - 2pzt + 2p^2z^2 - pz^2 = 0$, given by

$$\begin{aligned}t &= \frac{2pz + \sqrt{4p^2z^2 - 8p^2z^2 + 4p^2z^2}}{2} = pz + zp^{\frac{1}{2}}(1 - p)^{\frac{1}{2}} \\ &= t_0(1 + \sqrt{0_A})\end{aligned}\quad (\text{A.3})$$

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